Occurence and non-appearance of shocks in fractal Burgers equations

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Abstract We consider the fractal Burgers equation (that is to say the Burgers equation to which is added a fractional power of the Laplacian) and we prove that, if the power of the Laplacian involved is lower than 1/2, then the equation does not regularize the initial condition: on the contrary to what happens if the power of the Laplacian is greater than 1/2, discontinuities in the initial data can persist in the solution and shocks can develop even for smooth initial data. We also prove that the creation of shocks can occur only for sufficiently "large" initial conditions, by giving a result which states that, for smooth "small" initial data, the solution remains at least Lipschitz continuous.

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1 Introduction and main results

We consider the fractal Burgers equation

$$\partial_t u(t,x) + \partial_x \left(\frac{1}{2}u^2\right)(t,x) + g[u(t,\cdot)](x) = 0, \qquad (t,x) \in]0,+\infty[\times\mathbb{R}, \tag{1.1}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where u_0 is bounded and g is the non-local operator defined through the Fourier transform by

$$\mathcal{F}(g[\varphi])(\xi) = |\xi|^\lambda \mathcal{F}(\varphi)(\xi) \quad \text{with } \lambda \in]0,1[,$$

i.e. g is the fractional power of order $\lambda/2$ of the Laplacian.

This equation is involved in many different physical problems, such as overdriven detonation in gas [6] or anomalous diffusion in semiconductor growth [12], and has been studied in a number of papers, such as [1, 3, 4, 7, 8].
It is well known that the pure Burgers equation (i.e. (1.1) without $g[u]$) can give rise to shocks: even for some smooth initial data, the solution can become discontinuous in finite time. On the other hand, the parabolic regularization of the Burgers equation (i.e. (1.1) with $\lambda = 2$, that is to say $g[u] = -\Delta u$ up to a positive multiplicative constant depending on the definition of the Fourier transform) avoids such situations and has smooth solutions, even for merely bounded initial data. It has been proved in [7] that, if $\lambda > 1$, then (1.1) has the same behaviour as the parabolic regularization: for any bounded initial data, the solution is smooth. If $\lambda \le 1$, the regularity of the solution is not completely clear; for example, it is shown in [3] that, for $\lambda \in]1/2, 1]$, small initial data in $H^1(\mathbb{R})$ give rise to solutions which remain in $H^1(\mathbb{R})$, and insights are given as to why, if the initial data is not small, the solution may exhibits shocks (but no proof of this fact is made: the insight for the creation of shocks is just that, if $\lambda < 1$, no bounded traveling wave solution exist). To our best knowledge, there does not exist any proof that smooth initial data can give rise to discontinuous solutions to (1.1) if $\lambda \in]0, 1]$.

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One of the difficulties to study (1.1)-(1.2) for $\lambda \leq 1$ is that uniqueness of weak solutions is not obvious (precisely because they lack regularity); if the initial data is regular and small enough, some uniqueness results of the weak solution exist in [3], but for general bounded initial data, one has to use the notion of entropy solution developped in [1] in order to ensure existence and uniqueness of the (possibly irregular) solution. The question which interests us here is the following: is this solution really irregular? For smooth initial data, does (1.1) create shocks?

It is quite simple to see that the pure fractal equation $\partial_t v + g[v] = 0$ has, even for $\lambda \leq 1$, a regularizing effect: bounded initial data give rise to smooth solutions (see the properties of the kernel of g in Section 2.3 below). Hence, if shocks occur in (1.1), they result from the hyberbolic part of the equation; since the Burgers equation gives rise to shocks only for initial data which are somewhere decreasing, these are the ones we must consider in order to observe shocks in the solution to (1.1) (in fact, from the splitting method used in [7, 1], it is easy to see that, for non-decreasing smooth initial data, the solution to (1.1) remains Lipschitz continuous).
Our main assumption on the initial data is the following:

$$u_0 : \mathbb{R} \to \mathbb{R} \text{ is bounded, odd on } \mathbb{R} \text{ and convex on } \mathbb{R}^+ \tag{1.3}$$

(notice that u_0 is then locally Lipschitz continuous, non-increasing and non-positive on $\mathbb{R}^+_*$). These initial data can be smooth on $\mathbb{R}$ or discontinuous at $x = 0$. One can remark that the Riemann initial condition which gives rise to an entropy shock for the Burgers equation, i.e. $u_0(x) = +1$ if $x < 0$ and $u_0(x) = -1$ if $x > 0$, satisfies (1.3).
The fractal and hyperbolic operators in (1.1) are then competitors: the first one tends to regularize the solution, whereas the second one tends to create shocks. We will indeed light up this competition, by showing that, depending on the "size" of the initial data, in some cases the hyperbolic operator dominates and shocks occur, whereas in some cases the regularizing effect is stronger and the solution remains Lipschitz continuous. Let us now precisely describe our results.

The first theorem states that an initial discontinuity cannot instantly disappear (the operator g is not regularizing enough if $\lambda < 1$).

Theorem 1.1 (Preservation of initial shock) *Let $\lambda \in]0, 1[$. Assume that u_0 satisfies (1.3) and is discontinuous at $x = 0$. Then, for small times, the unique entropy solution u to (1.1)-(1.2) (see Definition 2.1) remains discontinuous along the axis $\{x = 0\}$.*
More precisely, $u \in C_b([0, +\infty[\times\mathbb{R}_*)$ is odd and non-increasing with respect to the space variable and there exist $\varepsilon > 0$ such that

$$\inf_{t\in[0,0+\varepsilon[} \{u(t, 0^-) - u(t, 0^+)\} > 0. \tag{1.4}$$

where $u(t, 0^\pm)$ denote the limits $\lim_{x\to 0^\pm} u(t, x)$.

The second result is somewhat stronger, since it shows that, for some *smooth* initial data, a shock occurs in the solution.

Theorem 1.2 (Creation of shock) *Let $\lambda \in]0, 1[$. There exists $S(\lambda) > 0$ such that, if u_0 satisfies (1.3) and*

$$\exists x_* > 0 \text{ such that } u_0(x_*) < -S(\lambda)x_*^{1-\lambda}\,, \tag{1.5}$$

then the unique entropy solution u to (1.1)-(1.2) (see Definition 2.1) develops a line of discontinuities in finite time along the axis $\{x = 0\}$.
More precisely, $u \in C_b([0, +\infty[\times\mathbb{R}_*)$ is odd and non-increasing with respect to the space variable and there exist $0 \leq t_* < +\infty$ and $\varepsilon > 0$ such that

$$\inf_{t\in[t_*,t_*+\varepsilon[} \{u(t, 0^-) - u(t, 0^+)\} > 0, \tag{1.6}$$

where $u(t, 0^\pm)$ denote the limits $\lim_{x\to 0^\pm} u(t, x)$.

Remark 1.3 *The proof shows that one can take* $S(\lambda)=\frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}$ *(where* G_λ *is defined by (2.2)) and that the shock occurs before the time* $t=\frac{x_*}{-u_0(x_*)-S(\lambda)x_*^{1-\lambda}}$.

In the last result, we state a counterpart of Theorem 1.2: if the initial data and its derivative are not simultaneously large, then no shock is created and the solution remains at least Lipschitz continuous.

Theorem 1.4 (No creation of shock) *Let* $\lambda\in]0,1[$. *Define* G_λ *by (2.2) and take* $L>0$ *and* $M>0$ *such that*

$$L^{1-\lambda}M^\lambda<\frac{G_\lambda}{2^\lambda\lambda}. \tag{1.7}$$

If $u_0\in W^{1,\infty}(\mathbb{R})$ *satisfies (1.3),* $||u_0||_{L^\infty(\mathbb{R})}\le M$ *and* $||u_0'||_{L^\infty(\mathbb{R})}\le L$, *then the entropy solution* u *to (1.1)-(1.2) (see Definition 2.1) belongs to* $W^{1,\infty}(]0,+\infty[\times\mathbb{R})$ *and satisfies* $||u(t,\cdot)||_{L^\infty(\mathbb{R})}\le M$ *and* $||\partial_x u(t,\cdot)||_{L^\infty(\mathbb{R})}\le L$ *for all* $t\ge 0$.

It is easy to check that if $u_0\in W^{1,\infty}(\mathbb{R})$ satisfies (1.5) with $S(\lambda)$ as in Remark 1.3 and if $u_0(0)=0$ (which is the case if (1.3) holds), then $M=||u_0||_{L^\infty(\mathbb{R})}$ and $L=||u_0'||_{L^\infty(\mathbb{R})}$ cannot satisfy (1.7) ([1]). Notice however that, for all $A>0$ and all $S>0$, there exists $u_0\in W^{1,\infty}(\mathbb{R})$ which satisfies (1.3) and such that $u_0(x)\ge -Sx^{1-\lambda}$ for all $x\in\mathbb{R}$ and $||u_0'||^{1-\lambda}_{L^\infty(\mathbb{R})}||u_0||^\lambda_{L^\infty(\mathbb{R})}\ge A$ (i.e. the opposites of (1.5) and (1.7) simultaneously hold, with free constants).
Relations (1.5) and (1.7) therefore are two "ordered" thresholds on the relative sizes of the initial data and its derivative; under the lower threshold (1.7), the solution to (1.1)-(1.2) remains Lipschitz continuous and, above the upper threshold (1.5), this solution develops shocks. For initial data which are between the two thresholds, it is not clear if shocks occur or not. Our results are thus of the same kind as in [11], where two such thresholds are given in the case where $g[u]$ in (1.1) is replaced by a zero-order convolution term.

The paper is organized as follows. In Section 2, we recall some basic facts about fractal operators and fractal conservation laws. Section 3 is devoted to the proof of Theorems 1.1 and 1.2: we first show that the fractal Burgers equation preserves (1.3) (if the initial data satisfies this property, then the solution too), and we then introduce a method of characteristics for (1.1) which allows to prove the theorems. In Section 4, we prove Theorem 1.4 by showing that, during the splitting method which consists in separately solving the Burgers equation and the fractal equation, the fractal equation "compensates" the tendency of the Burgers equation to create shocks. We have gathered some simple technical lemmas, used throughout the paper, in an appendix in Section 5.

2 Preliminary results

We recall here some facts concerning the fractal operator g and the associated equations.

2.1 Integral representation of g

It is proved (in [8] for example) that, if $\lambda\in]0,1[$, the operator g can be written in another way: for all Schwartz function φ, we have

$$g[\varphi](x)=-G_\lambda\int_{\mathbb{R}}\frac{\varphi(x+z)-\varphi(x)}{|z|^{1+\lambda}}\,dz \tag{2.1}$$

where

$$G_\lambda=\frac{\lambda\Gamma(\frac{1+\lambda}{2})}{2\pi^{\frac{1}{2}+\lambda}\Gamma(1-\frac{\lambda}{2})}>0 \tag{2.2}$$

[1]Indeed, from (1.5) we clearly have $M\ge S(\lambda)x_*^{1-\lambda}$ and $L\ge\frac{|u_0(x_*)-u_0(0)|}{x_*}\ge S(\lambda)x_*^{-\lambda}$, so that $L^{1-\lambda}M^\lambda\ge S(\lambda)=\frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}=\frac{2}{(1-\lambda)^2}\frac{G_\lambda}{2^\lambda\lambda}>\frac{G_\lambda}{2^\lambda\lambda}$.

(Γ is Euler's function). Thanks to this formula, we can consider that g is an operator $W^{1,\infty}(\mathbb{R}) \to C_b(\mathbb{R})$ and $C_b(\mathbb{R}_*) \cap W^{1,\infty}_{\rm loc}(\mathbb{R}_*) \to C(\mathbb{R}_*)$.

2.2 Entropy solutions for fractal conservation laws

If one considers the general fractal conservation law

$$\partial_t u(t,x) + \partial_x(f(u))(t,x) + g[u(t,\cdot)](x) = 0, \qquad (t,x) \in]0,+\infty[\times\mathbb{R}, \tag{2.3}$$

$$u(0,x) = u_0(x), \qquad x \in \mathbb{R}, \tag{2.4}$$

with $f:\mathbb{R}\to\mathbb{R}$ locally Lipschitz continuous, the integral representation (2.1) of g motivates the following definition, from [1], of entropy solutions to (2.3)-(2.4).

Definition 2.1 (Entropy solution) *Let $\lambda \in]0,1[$ and $u_0 \in L^\infty(\mathbb{R})$. An entropy solution to (2.3)-(2.4) is a function $u \in L^\infty(]0,+\infty[\times\mathbb{R})$ such that, for all non-negative $\varphi \in C_c^\infty([0,+\infty[\times\mathbb{R})$, for all smooth convex function $\eta:\mathbb{R}\to\mathbb{R}$, all $\phi:\mathbb{R}\to\mathbb{R}$ such that $\phi' = \eta' f'$ and all $r>0$, we have*

$$\int_0^\infty \int_{\mathbb{R}} (\eta(u)\partial_t\varphi + \phi(u)\partial_x\varphi) + G_\lambda \int_0^\infty \int_{\mathbb{R}} \int_{|z|>r} \eta'(u(t,x)) \frac{u(t,x+z)-u(t,x)}{|z|^{1+\lambda}} \varphi(t,x)\,dtdxdz$$
$$+ G_\lambda \int_0^\infty \int_{\mathbb{R}} \int_{|z|\le r} \eta(u(t,x)) \frac{\varphi(t,x+z)-\varphi(t,x)}{|z|^{1+\lambda}}\,dtdxdz + \int_{\mathbb{R}} \eta(u_0)\varphi(0,\cdot) \ge 0.$$

Remark 2.2 *(see [1]) This definition can be extended to the case $\lambda = 1$ and to multidimensional equations, and provides existence and uniqueness of the solution to (2.3)-(2.4); moreover, this entropy solution is bounded by $||u_0||_{L^\infty(\mathbb{R})}$.*

The entropy solution to (2.3)-(2.4) can by constructed by using a splitting method.

Splitting method (see [1, 7]): for $\delta > 0$, we construct $u^\delta : [0,+\infty[\times\mathbb{R} \to \mathbb{R}$ the following way: we let $u^\delta(0,\cdot) = u_0$ and, for all even p and all odd q, we define by induction

(a) u^δ on $]p\delta,(p+1)\delta]\times\mathbb{R}$ as the solution to $\partial_t u^\delta + 2g[u^\delta] = 0$ with initial condition $u^\delta(p\delta,\cdot)$ (that is to say $u^\delta(t,x) = K(2(t-p\delta),\cdot) * u^\delta(p\delta,\cdot)(x)$ where K is the kernel of g, see Section 2.3);

(b) u^δ on $]q\delta,(q+1)\delta]\times\mathbb{R}$ as the entropy solution to $\partial_t u^\delta + 2\partial_x(f(u^\delta)) = 0$ with initial condition $u^\delta(q\delta,\cdot)$.

As proved in [1], the function u^δ thus constructed converges, as $\delta\to 0$ and in $C([0,T];L^1_{\rm loc}(\mathbb{R}))$ for all $T>0$, to the unique entropy solution to (2.3)-(2.4). This is the only fact we will need concerning entropy solutions to (2.3)-(2.4).

2.3 Kernel of g

The Fourier transform shows that the solution to $\partial_t v + g[v] = 0$ with initial condition v_0 is given by $v(t,x) = K(t,\cdot) * v_0(x)$ where

$$K(t,\cdot) = \mathcal{F}^{-1}(e^{-t|\cdot|^\lambda}).$$

It can be shown (see e.g. [9, 7]) that the kernel of g satisfies the following properties ([2]).

$$\begin{gathered} K(1,\cdot) \in C_b^\infty(\mathbb{R}) \cap W^{\infty,1}(\mathbb{R}) \text{ is even and non-negative}, \\ \int_{\mathbb{R}} K(1,x)\,dx = 1, \\ K(t,x) = t^{-\frac{1}{\lambda}} K(1, t^{-\frac{1}{\lambda}} x). \end{gathered} \tag{2.5}$$

[2]In fact, the integrability of the *derivatives* of $K(1,\cdot)$ can be obtained by proving, as in [7], that they all are $\mathcal{O}(1/(1+|\cdot|^2))$, but, because $\lambda < 1$, the integrability of $K(1,\cdot)$ itself cannot be deduced the same way. To see that $K(1,\cdot) \in L^1(\mathbb{R})$, one can invoke the fact that the sequence $(f_n)_{n\ge 1}$ from the proof of Lemma 2.3 is bounded in $L^1(\mathbb{R})$ and converges in $\mathcal{S}'(\mathbb{R})$ to $K(c,\cdot)$ for some $c>0$, so that $K(c,\cdot)$ is necessarily a bounded measure on $\mathbb{R}$: since it is a function, this shows that it is integrable, and hence $K(1,\cdot)$ also by homogeneity.

Another important feature of K is the following.

Lemma 2.3 *If $\lambda \in]0,2]$, then, for all $t>0$, $K(t,.)$ is non-increasing on $\mathbb{R}^+$.*

Proof of Lemma 2.3
For $\lambda = 2$ it is well-known that K is a Gaussian function, which implies the result. Assume now that $\lambda \in]0,2[$. The non-negativity of the kernel can be proved by approximating K by a sequence of functions known to be non-negative (see [7, Lemma 2.1]). To prove that $K(t,\cdot)$ is non-increasing on $\mathbb{R}^+$, we slightly modify this sequence of functions so that they are also non-increasing on $\mathbb{R}^+$ (and, in fact, the proof that follows also shows that $K \geq 0$).
Let

$$f(x) = A\left(|x|^{-1-\lambda}\mathbf{1}_{\mathbb{R}\setminus]-1,1[}(x) + \mathbf{1}_{]-1,1[}(x)\right),$$

with $A>0$ such that $\int_{\mathbb{R}} f = 1$. Since f is even with integral equal to 1, we have

$$\begin{aligned}\mathcal{F}(f)(\xi) &= 1 + \int_{\mathbb{R}} (\cos(2\pi x\xi) - 1) f(x)dx \\ &= 1 + A|\xi|^{\lambda}\int_{|y|\geq|\xi|} \frac{\cos(2\pi y) - 1}{|y|^{1+\lambda}}dy + A|\xi|^{-1}\int_{|y|\leq|\xi|}(\cos(2\pi y) - 1)dy.\end{aligned}$$

Since $\cos(2\pi y) - 1 = \mathcal{O}(|y|^2)$ on the neighborhood of 0, the last term of this inequality equals $\mathcal{O}(|\xi|^2)$. Moreover, as $\lambda < 2$, the dominated convergence theorem gives

$$\int_{|y|\geq|\xi|} \frac{\cos(2\pi y) - 1}{|y|^{1+\lambda}}dy \to I := \int_{\mathbb{R}} \frac{\cos(2\pi y) - 1}{|y|^{1+\lambda}}dy < 0 \quad \text{as } \xi \to 0.$$

Then $\mathcal{F}(f)(\xi) = 1 - c|\xi|^{\lambda}(1+\omega(\xi))$ with $c = -AI > 0$ and $\lim_{\xi\to 0}\omega(\xi) = 0$. Define $f_n(x) = n^{1/\lambda} f * \cdots * f(n^{1/\lambda}x)$, the convolution product being taken n times. By the properties of Fourier transform with respect to the convolution product, we have, for all $\xi \in \mathbb{R}$,

$$\mathcal{F}(f_n)(\xi) = \left(\mathcal{F}(f)(n^{-1/\lambda}\xi)\right)^n = \left(1 - cn^{-1}|\xi|^{\lambda}(1+\omega(n^{-1/\lambda}\xi))\right)^n \to e^{-c|\xi|^{\lambda}} \quad \text{as } n \to +\infty.$$

Since $(\mathcal{F}(f_n))_{n\geq 1}$ is bounded by 1 (the L^1 norm of f_n for all $n \geq 1$), this convergence also holds in $\mathcal{S}'(\mathbb{R})$. Taking the inverse Fourier transform, we see that $f_n \to K(c,.)$ in $\mathcal{S}'(\mathbb{R})$ as $n \to +\infty$.
The function $f \in L^1(\mathbb{R})$ is even, non-negative on $\mathbb{R}$ and non-increasing on $\mathbb{R}^+$. Arguing by induction, Lemma 5.1 in the appendix allows to prove that f_n also satisfies these properties; it is quite simple to see that the convergence in $\mathcal{S}'(\mathbb{R})$ preserves these properties, which shows in particular that $K(c,.)$ is non-increasing on $\mathbb{R}^+$. By the homogeneity property of K mentioned in (2.5), the proof of the lemma is complete. ■

3 Preservation and creation of shock

This section is devoted to the proofs of Theorems 1.1 and 1.2.

3.1 Property (1.3) is preserved

The following result is central in the study of (1.1)-(1.2) for initial data which satisfy (1.3).

Lemma 3.1 *Let $\lambda \in]0,1[$, u_0 satisfy (1.3) and u be the entropy solution to (1.1)-(1.2). Then $u \in C_b([0,+\infty[\times\mathbb{R}_*) \cap W^{1,\infty}_{\mathrm{loc}}([0,+\infty[\times\mathbb{R}_*)$ and, for all $t>0$, $u(t,\cdot)$ satisfies (1.3).*

Remark 3.2 *This lemma is also true for $\lambda \in [1,2]$, but will not be useful to us in this setting.*

Proof of Lemma 3.1

The idea is to use the splitting method, as described in Section 2.2, by proving that both equations $\partial_t u + 2g[u] = 0$ and $\partial_t u + 2\partial_x(\frac{1}{2}u^2) = 0$ preserve (1.3).

Step 1: conservation of (1.3) by the fractal equation.

Assume that u_0 satisfies (1.3). Since u_0 is locally Lipschitz continuous on $\mathbb{R}_*$, it has a classical derivative $u_0' \in L^\infty_{\mathrm{loc}}(\mathbb{R}_*)$ and, for $y > x > 0$, we have

$$2||u_0||_{L^\infty(\mathbb{R})} \geq u_0(x) - u_0(y) = \int_x^y -u_0'(s)\,ds.$$

But $-u_0' \geq 0$ on $\mathbb{R}_*^+$ (because u_0 is non-increasing on this set, by (1.3)) and, letting $x \to 0$ and $y \to \infty$, we obtain

$$\int_0^\infty |u_0'(s)|\,ds \leq 2||u_0||_{L^\infty(\mathbb{R})}.$$

Since u_0 is odd, this proves that $u_0' \in L^1(\mathbb{R})$. From this, and denoting $J = u_0(0^+) - u_0(0^-) \leq 0$ the jump of u_0 at $x = 0$, it is easy to see that the distributional derivative of u_0 on $\mathbb{R}$ is $Du_0 = u_0' + J\delta_0$.

Define now $u(t,.) := K(2t,.) * u_0$ for $t > 0$ (i.e. u is the solution to $\partial_t u + 2g[u] = 0$ with initial data u_0). By the properties of K, u is a well-defined, bounded (by $||u_0||_{L^\infty(\mathbb{R})}$) and smooth function (see also [7] or [8, Lemma 2]). Since $K(2t,\cdot)$ is even and u_0 is odd, it is quite obvious that $u(t,\cdot)$ is odd. Moreover, as $Du_0 = u_0' + J\delta_0$, it is easy to see that $\partial_x u(t,\cdot) = K(2t,\cdot) * u_0' + JK(2t,\cdot)$. By (1.3), we see that u_0' is even, non-positive on $\mathbb{R}$ and non-decreasing on $\mathbb{R}^+$. From Lemma 2.3, Property (2.5), Lemma 5.1 (applied to $-u_0'$) and the fact that $J \leq 0$, we deduce that $\partial_x u(t,\cdot)$ is non-decreasing on $\mathbb{R}^+$, and therefore that $u(t,\cdot)$ is convex on $\mathbb{R}^+$. Hence, $u(t,\cdot)$ is a smooth function which satisfies (1.3).

Step 2: conservation of (1.3) by the Burgers equation.

Let us resolve the Burgers equation $\partial_t u + 2\partial_x(\frac{u^2}{2}) = 0$ by the classical method of characteristics. We assume here that u_0 is smooth (as we will see, this is not a loss of generality) and satisfies (1.3).

The characteristics for the Burgers equation with initial datum u_0 are $t \to x_0 + 2tu_0(x_0)$. Since u_0 is odd, the characteristics from x_0 and $-x_0$ are symmetric with respect to $x = 0$; in fact, by (1.3) we see that u_0 is negative on $\mathbb{R}_*^+$ and positive on $\mathbb{R}_*^-$ (unless it vanishes on $\mathbb{R}$, a case where the conservation of property (1.3) by the Burgers equation is obvious), and since u_0 is non-increasing, the characteristics behave as in Figure 1.

As suggested by this figure, we can prove that the characteristics coming from points $x_0 > 0$ form a partition of $[0,+\infty[\times\mathbb{R}_*^+$: they do not intersect and cover this whole domain. Indeed, for $x > 0$ consider $h(x) = -\frac{x}{2u_0(x)} > 0$, the point on the t-axis where the characteristic $t \to x + 2tu_0(x)$ crosses this axis (recall that, unless it completely vanishes, $u_0(x) < 0$ for all $x > 0$). We have $\mathrm{sgn}(h'(x)) = \mathrm{sgn}(xu_0'(x) - u_0(x))$ and, since (1.3) implies ([3])

$$u_0(x) \leq x\,u_0'(x) \quad \text{for all } x > 0, \tag{3.1}$$

we deduce that h is non-decreasing on $\mathbb{R}_*^+$. Hence, the only point where two characteristics originating for $x_0 > 0$ and $y_0 > 0$ can intersect is at $x = 0$, and not in $[0,+\infty[\times\mathbb{R}_*^+$. Let $t \geq 0$ and $y > 0$; the continuous function $x \to x + 2tu_0(x)$ is equal to 0 at $x = 0$ (because $u_0(0) = 0$ by (1.3)) and, since u_0 is bounded, has limit $+\infty$ as $x \to +\infty$; hence, there exists $x > 0$ such that $x + 2tu_0(x) = y$, which shows that the characteristics cover the whole domain $[0,+\infty[\times\mathbb{R}_*^+$.

This proves that, in the domain $[0,+\infty[\times\mathbb{R}_*^+$, the solution u to the Burgers equation stays smooth and can be computed thanks to the characteristics. Let $t > 0$ and $x_0 > 0$ such that $t < h(x_0)$ (i.e. $(t, x_0 + 2tu_0(x_0)) \in [0,+\infty[\times\mathbb{R}_*^+$); we have $u(t, x_0 + 2tu_0(x_0)) = u_0(x_0)$ and we can differentiate with respect to x_0 (the x_0 such that $t < h(x_0)$ form an open set) to find

$$\partial_x u(t, x_0 + 2tu_0(x_0))(1 + 2tu_0'(x_0)) = u_0'(x_0). \tag{3.2}$$

[3] This is the classical slopes inequality for convex functions, between the points $(0,0) = (0,u_0(0))$ and $(x,u_0(x))$.

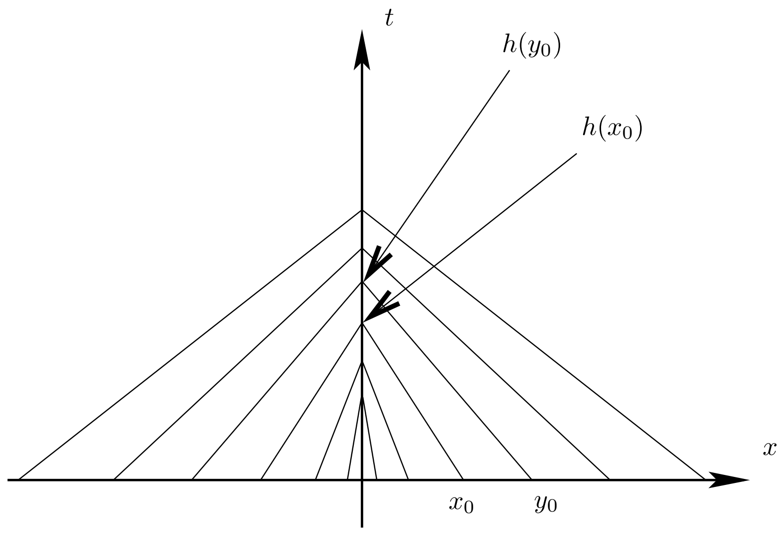

Figure 1: Characteristics for the Burgers equation in the case where u_0 satisfies (1.3).

By (3.1), we have $1+2tu_0'(x_0)\ge 1-\frac{t}{h(x_0)}>0$; hence, (3.2) shows that

$$\partial_x u(t,x_0+2tu_0(x_0))=\frac{u_0'(x_0)}{1+2tu_0'(x_0)}. \tag{3.3}$$

Let $t\ge 0$, $x>y>0$ and take $x_0>0$ and $y_0>0$ such that $t<h(x_0)$, $t<h(y_0)$, $x_0+2tu_0(x_0)=x$ and $y_0+2tu_0(y_0)=y$ (this is possible because the characteristics originating from positive points cover $[0,+\infty[\times\mathbb{R}^+_*$). The preceding reasoning shows that $z\to z+2tu_0(z)$ is increasing on the interval $\{z>0\mid t<h(z)\}$ (its derivative $1+2tu_0'(z)$ is positive), and therefore $x_0>y_0$. Since u_0' is non-decreasing on $\mathbb{R}^+_*$ and $p\to\frac{p}{1+2tp}$ is non-decreasing on the interval $\{p\mid 1+2tp>0\}$, we deduce from (3.3) applied to x_0 and y_0 that $\partial_x u(t,x)\ge\partial_x u(t,y)$, and therefore that $u(t,\cdot)$ is convex on $\mathbb{R}^+_*$. Since u is obviously odd with respect to the space variable (because $-u(\cdot,-\cdot)$ is another entropy solution of the Burgers equation, and is therefore equal to u) and non-positive on $[0,+\infty[\times\mathbb{R}^+_*$ (the characteristics show that the values of u on this set are given by values of u_0 on $\mathbb{R}^+_*$), the convexity of $u(t,\cdot)$ on $\mathbb{R}^+_*$ entails its convexity on $\mathbb{R}^+$ (u is null at $x=0$), and this concludes the proof that, if the initial datum is regular and satisfies (1.3), then the solution of the Burgers equation is regular in $[0,+\infty[\times\mathbb{R}_*$ and satisfies (1.3) at any time.

Step 3: conclusion.
Consider now u_0 bounded which satisfies (1.3). For $\delta>0$, construct u^δ using the splitting method presented in Section 2.2. By the preceding steps, we know that $u^\delta\in C_b([0,+\infty[\times\mathbb{R}_*)\cap W^{1,\infty}_{\rm loc}([0,+\infty[\times\mathbb{R}_*)$ (it stays smooth outside $x=0$) and that, for all $t\ge 0$, $u^\delta(t,.)$ satisfies (1.3) (notice that, in the splitting method, all the Burgers problems we solve have regular initial data, coming from the resolution of the fractal equation at the preceding time step).
Since u^δ is also bounded independently of δ (by $||u_0||_{L^\infty(\mathbb{R})}$, as the fractal and the hyperbolic equations do not increase the L^∞ norm), we deduce from its convexity properties that $u^\delta(t,\cdot)$ is locally Lipschitz continuous on $\mathbb{R}_*$ with local Lipschitz constants which do not depend on t or δ. Using the hyperbolic and the fractal equations to then control the time derivative of u^δ (if one controls the $L^\infty(\mathbb{R})$ norm and local Lipschitz constants of φ, then (2.1) shows that one has local bounds on $g[\varphi]$), we deduce that u^δ is in fact locally Lipschitz continuous on $[0,+\infty[\times\mathbb{R}_*$ with local Lipschitz constants which do not depend on $\delta>0$. By the Ascoli-Arzela theorem, the family $\{u^\delta:\delta>0\}$ is relatively compact in $C([0,T]\times Q)$, for all $T>0$ and all Q compact subset of $\mathbb{R}_*$. Since u^δ converges in $C([0,T];L^1_{\rm loc}(\mathbb{R}))$, for all $T>0$ and

as $\delta \to 0$, to the entropy solution u to (1.1)-(1.2), we deduce that u^δ converges to u locally uniformly on $[0,+\infty[\times\mathbb{R}_*$ and that u is locally Lipschitz continuous on $[0,+\infty[\times\mathbb{R}_*$.
The proof is then concluded by recalling that, for all $t \geq 0$, $u^\delta(t,\cdot)$ satisfies (1.3) and converges locally uniformly to $u(t,\cdot)$ on $\mathbb{R}_*$, which implies that $u(t,\cdot)$ also satisfies (1.3) ([4]). ■

3.2 The generalized characteristic method

In the following, we take u_0 which satisfies (1.3) and we denote u the entropy solution to (1.1)-(1.2). By the regularity of u in Lemma 3.1 and the Cauchy-Lipschitz theorem, for all $x_0 \in \mathbb{R}_*$ there exists a unique maximal solution $x : I_{x_0} \subset [0,+\infty[\to \mathbb{R}_*$ to

$$\left\{ \begin{array}{ll} x'(t) = u(t,x(t)) \,, & t \in I_{x_0} \,, \\ x(0) = x_0. & \end{array} \right. \tag{3.4}$$

Notice that, since we are not sure that u is regular at $x = 0$, it is natural to consider only solutions with values in $\mathbb{R}_*$, and the maximality property is subordinate to this condition $x(t) \in \mathbb{R}_*$.

Definition 3.3 *Let $\lambda \in]0,1[$, u_0 satisfy (1.3) and u be the entropy solution to (1.1)-(1.2). A generalized characteristic of (1.1)-(1.2) originating from $x_0 \in \mathbb{R}_*$ is the maximal solution $x : I_{x_0} \to \mathbb{R}_*$ to (3.4).*

Remark 3.4 *Since u is odd with respect to the space variable, the graphs of the generalized characteristics originating for x_0 and $-x_0$ are, as in the case of pure Burgers equation, symmetric with respect to $x = 0$.*

Let us first give some basic properties on the generalized characteristics.

Lemma 3.5 *Let $\lambda \in]0,1[$, u_0 satisfy (1.3) and u be the entropy solution to (1.1)-(1.2). The generalized characteristics satisfy the following properties.*

i) The graphs of the generalized characteristics originating for points $x_0 > 0$ form a partition of $[0,+\infty[\times\mathbb{R}_*^+$.

ii) Let $x_0 \in \mathbb{R}_*$ and $x(t)$ be the generalized characteristic originating from x_0. If $I_{x_0} \neq [0,+\infty[$, then $I_{x_0} = [0,t_*[$ with $t_* < +\infty$ and $\lim_{t\to t_*^-} x(t) = 0$.

iii) u is continuously derivable along the generalized characteristics and, for all generalized characteristic $x : I_{x_0} \to \mathbb{R}_*$,

$$\frac{d}{dt}u(t,x(t)) = -g[u(t,\cdot)](x(t)) \quad \textit{for all } t \in I_{x_0}.$$

Proof of Lemma 3.5
For classical results on ODE that are used during the proof, we refer the reader to [5] or [2].
Let us first prove Item i). For $(t_0,y_0) \in [0,+\infty[\times\mathbb{R}_*^+$, we consider the maximal solution $y : I_{t_0,y_0} \to \mathbb{R}_*^+$ to the Cauchy problem

$$\left\{ \begin{array}{ll} y'(t) = u(t,y(t)), & t \in I_{t_0,y_0} \,, \\ y(t_0) = y_0 & \end{array} \right. \tag{3.5}$$

and we want to show that $0 \in I_{t_0,y_0}$. Since u is non-positive on $[0,+\infty[\times\mathbb{R}_*^+$, we have $0 \geq y'(t) \geq -||u_0||_{L^\infty(\mathbb{R})}$ for all $t \in I_{t_0,y_0}$. By integrating, we deduce that y is non-increasing on I_{t_0,y_0} and bounded from above by $y_0 + ||u_0||_{L^\infty(\mathbb{R})}(t_0 - t)$ for $t \in I_{t_0,y_0} \cap [0,t_0]$. The limit $\lim_{t \geq \inf I_{t_0,y_0}} y(t) = x_0$ then exists and belongs to $[y_0, y_0 + ||u_0||_{L^\infty(\mathbb{R})}t_0] \subset \mathbb{R}_*^+$. By maximality of y, this means that $\inf I_{t_0,y_0} = 0$ (or else y can be extended beyond this infimum, since u is locally Lipschitz continuous on $[0,+\infty[\times\mathbb{R}_*^+$) and that y is equal to the generalized characteristic x originating from x_0. Hence, the graphs of the generalized

[4] $u(t,\cdot)$ is not necessarily well defined at $x = 0$ but, in this case, we of course take the representative of $u(t,\cdot)$ which satisfies $u(t,0) = 0$.

characteristics originating from points in $\mathbb{R}_*^+$ cover the whole domain $[0,+\infty[\times\mathbb{R}_*^+$. The Cauchy-Lipschitz theorem ensures that these graphs never cross, and this concludes the proof of i).

Item ii) is easy to prove. Indeed, let $x_0>0$ (by symmetry, there is no loss of generality in assuming this) and suppose that the suppremum of the interval I_{x_0} is $t_*<+\infty$; since u is non-positive on $[0,+\infty[\times\mathbb{R}_*^+$, we have $0\geq x'(t)$ so that x is non-increasing and has a limit in $[0,x_0]$ as $t\to t_*^-$. If this limit is positive, then u being locally Lipschitz continuous on $[0,+\infty[\times\mathbb{R}_*^+$ (see Lemma 3.1), the maximal solution $x(t)$ to (3.4) could be extended beyond the time t_*, which is a contradiction; hence, $\lim_{t\to t_*^-}x(t)=0$.

Let us now prove Item iii). Let $\mathcal{U}=\{(t,x_0)\in[0,+\infty[\times\mathbb{R}_*\,,\ t\in I_{x_0}\}$ and $\phi:(t,x_0)\in\mathcal{U}\to(t,x(t))\in[0,+\infty[\times\mathbb{R}_*$, where $x:I_{x_0}\to\mathbb{R}_*$ is the generalized characteristic originating from x_0. Classical results on ODE imply that $\mathcal{U}$ is an open subset of $[0,+\infty[\times\mathbb{R}_*$ and that ϕ is a locally Lipschitz continuous homeomorphism (it is bijective thanks to Item i) of the lemma and the symmetry of the characteristics with respect to $x=0$), derivable with respect to the time variable on $\mathcal{U}$ with $\partial_t(\pi\circ\phi)=u\circ\phi$, where π denotes the projection on the second factor of $[0,+\infty[\times\mathbb{R}_*$. By Lemma 3.1, u is locally Lipschitz continuous on $]0,+\infty[\times\mathbb{R}_*$ (and therefore a.e. derivable); the distributional derivatives of $u\circ\phi$ are thus equal to its a.e. derivatives, which can be computed by means of the chain rule since ϕ^{-1} preserves sets of null Lebesgue measure (it is locally Lipschitz continuous). Moreover, Lemma 3.1 and (2.1) imply that $g[u]\in C([0,+\infty[\times\mathbb{R}_*)$ and, since the entropy solution to (1.1)-(1.2) is also a weak solution (see [1]), this means that u satisfies (1.1) in the classical sense a.e. on $]0,+\infty[\times\mathbb{R}$. From all this we deduce, in the distributional sense,

$$\partial_t(u\circ\phi)=\partial_t u\circ\phi+(\partial_x u\circ\phi)(\partial_t(\pi\circ\phi))=\partial_t u\circ\phi+(\partial_x u\circ\phi)(u\circ\phi)=-g[u]\circ\phi. \tag{3.6}$$

Since $g[u]\circ\phi$ is continuous on $\mathcal{U}$, this implies that $u\circ\phi$ is in fact continuously derivable with respect to the time variable everywhere on $\mathcal{U}$, and (3.6) concludes the proof of the lemma. ■

Solutions to (3.4) are called "generalized characteristics" of (1.1)-(1.2) because, as in the case of pure scalar conservation law, we can establish some behaviour of the solution along these characteristics.

Lemma 3.6 *Let $\lambda\in]0,1[$, u_0 satisfy (1.3) and u be the entropy solution to (1.1)-(1.2). If $x_0>0$ and x is the generalized characteristic originating from x_0 then*

$$u(t,x(t))\leq u_0(x_0)+\frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}x_0^{1-\lambda}-\frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}x(t)^{1-\lambda}\qquad \textit{for all } t\in I_{x_0},$$

where G_λ is given by (2.2).

Proof of Lemma 3.6
By Item iii) in Lemma 3.5 and (2.1),

$$\frac{d}{dt}u(t,x(t))=G_\lambda\int_{\mathbb{R}}\frac{u(t,x(t)+z)-u(t,x(t))}{|z|^{1+\lambda}}dz.$$

Let us cut this integral term in three parts, according as $z<-2x(t)$, $-2x(t)\leq z\leq 0$ or $z>0$. We let P_1, P_2 and P_3 denote the respective parts, so that

$$\frac{d}{dt}u(t,x(t))=P_1+P_2+P_3. \tag{3.7}$$

By (3.4),

$$P_1 \;=\; G_\lambda\int_{-\infty}^{-2x(t)}\frac{u(t,x(t)+z)-u(t,x(t))}{|z|^{1+\lambda}}dz$$

$$
\begin{aligned}
&= G_\lambda \int_{-\infty}^{-2x(t)} \frac{-2u(t,x(t))}{|z|^{1+\lambda}} dz + G_\lambda \int_{-\infty}^{-2x(t)} \frac{u(t,x(t)+z)+u(t,x(t))}{|z|^{1+\lambda}} dz \\
&= -\frac{2^{1-\lambda} G_\lambda}{\lambda} \frac{x'(t)}{x(t)^\lambda} + G_\lambda \int_{-\infty}^{-2x(t)} \frac{u(t,x(t)+z)+u(t,x(t))}{|z|^{1+\lambda}} dz. \qquad (3.8)
\end{aligned}
$$

Let Q_1 denote this last integral term. Changing the variable by $z = -2x(t) - z'$ and since $u(t,\cdot)$ is odd (see Lemma 3.1), we get

$$
Q_1 = G_\lambda \int_0^{+\infty} \frac{u(t,-x(t)-z')+u(t,x(t))}{|2x(t)+z'|^{1+\lambda}} dz' = G_\lambda \int_0^{+\infty} \frac{-u(t,x(t)+z')+u(t,x(t))}{|2x(t)+z'|^{1+\lambda}} dz'.
$$

Using the fact that $u(t,.)$ is non-increasing on $\mathbb{R}_*^+$ and that $x(t) > 0$, we have $-u(t,x(t)+z')+u(t,x(t)) \geq 0$ and $|2x(t)+z'|^{-(1+\lambda)} \leq |z'|^{-(1+\lambda)}$ for all $z' > 0$. This implies $Q_1 + P_3 \leq 0$ and (3.8) gives

$$
P_1 + P_3 \leq -\frac{2^{1-\lambda} G_\lambda}{\lambda} \frac{x'(t)}{x(t)^\lambda}. \qquad (3.9)
$$

Moreover, since $\lambda < 1$ and still using (3.4),

$$
\begin{aligned}
P_2 &= G_\lambda \int_{-2x(t)}^{0} \frac{u(t,x(t)+z)-u(t,x(t))}{|z|^{1+\lambda}} dz \\
&= G_\lambda \int_{-2x(t)}^{0} \frac{\frac{u(t,x(t))}{x(t)} z}{|z|^{1+\lambda}} dz + G_\lambda \int_{-2x(t)}^{0} \frac{u(t,x(t)+z)-u(t,x(t)) - \frac{u(t,x(t))}{x(t)} z}{|z|^{1+\lambda}} dz \\
&= -\frac{2^{1-\lambda} G_\lambda}{1-\lambda} \frac{x'(t)}{x(t)^\lambda} + G_\lambda \int_{-2x(t)}^{0} \frac{u(t,x(t)+z)-u(t,x(t)) - \frac{u(t,x(t))}{x(t)} z}{|z|^{1+\lambda}} dz. \qquad (3.10)
\end{aligned}
$$

Let us cut the last integral sign in two pieces, according as $z < -x(t)$ or not; we let Q_2 and Q_3 denote the respective parts. We have, thanks to the change of variable $z = -2x(t) - z'$ and using the fact that $u(t,\cdot)$ is odd,

$$
\begin{aligned}
Q_2 &= G_\lambda \int_{-2x(t)}^{-x(t)} \frac{u(t,x(t)+z)-u(t,x(t)) - \frac{u(t,x(t))}{x(t)} z}{|z|^{1+\lambda}} dz \\
&= G_\lambda \int_{-x(t)}^{0} \frac{u(t,-x(t)-z')-u(t,x(t)) + \frac{u(t,x(t))}{x(t)} (2x(t)+z')}{|2x(t)+z'|^{1+\lambda}} dz' \\
&= G_\lambda \int_{-x(t)}^{0} \frac{-u(t,x(t)+z')+u(t,x(t)) + \frac{u(t,x(t))}{x(t)} z'}{|2x(t)+z'|^{1+\lambda}} dz'.
\end{aligned}
$$

Let $z' \in]-x(t), 0[$; the slopes inequality applied to the convex function $u(t,\cdot)$ (on $\mathbb{R}^+$) with the points $(0, u(t,0)) = (0,0)$, $(x(t)+z', u(t,x(t)+z'))$ and $(x(t), u(t,x(t))$ gives

$$
-u(t,x(t)+z') + u(t,x(t)) + \frac{u(t,x(t))}{x(t)} z' \geq 0. \qquad (3.11)
$$

If $-x(t) < z' < 0$ then $|z'| < x(t) < 2x(t)+z'$ and thus $|2x(t)+z'|^{-(1+\lambda)} \leq |z'|^{-(1+\lambda)}$; with (3.11), this gives $Q_2 + Q_3 \leq 0$. Inequality (3.10) then implies that $P_2 \leq -\frac{2^{1-\lambda} G_\lambda}{1-\lambda} \frac{x'(t)}{x(t)^\lambda}$ and, by (3.7) and (3.9), we deduce $\frac{d}{dt} u(t,x(t)) \leq -\frac{2^{1-\lambda} G_\lambda}{\lambda(1-\lambda)} \frac{x'(t)}{x(t)^\lambda}$ for $t \in I_{x_0}$. Integrating this inequality between 0 and t, the proof is complete. ■

3.3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1
Notice that $u \in C_b([0,+\infty[\times\mathbb{R}_*)$ is odd and non-increasing with respect to the space variable thanks to Lemma 3.1.
Let $S(\lambda) = \frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}$ (this is the $S(\lambda)$ of Remark 1.3). Since u_0 satisfies (1.3) and has a discontinuity at $x = 0$, we have $u_0(0^+) = -2\rho < 0$; there exists then $x_* > 0$ such that, for all $0 < x_0 \le x_*$, $u_0(x_0) + S(\lambda)x_0^{1-\lambda} \le u_0(0^+) + S(\lambda)x_*^{1-\lambda} \le -\rho$. The characteristic originating from x_* divides the space $I_{x_*} \times \mathbb{R}_*^+$ in two parts; we let E denote the left part (see Figure 2).

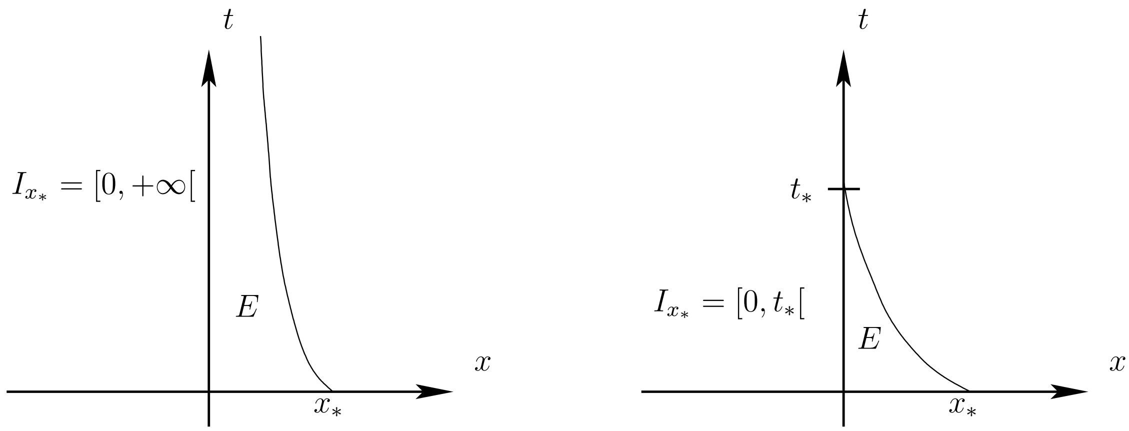

Figure 2: Division of the plane by the generalized characteristic originating from x_*, in the two possible cases $I_{x_*} = [0,+\infty[$ or $I_{x_*} = [0,t_*[$ with $t_* < +\infty$.

Item i) of Lemma 3.5 implies that E is included in the reunion of the graphs of the generalized characteristics originating from $0 < x_0 < x_*$ (in fact E is equal to this reunion) and from Lemma 3.6 and the choice of x_* we deduce that $\sup_E u \le -\rho$. For all $t \in I_{x_*}$, there exists $(t, y_n) \in E$ such that $y_n \to 0^+$ (because the generalized characteristic originating from x_* is positive at time t), and therefore $u(t,0^+) \le \sup_E u \le -\rho$. We therefore obtain $\sup_{t\in I_{x_*}} u(t,0^+) \le -\rho$ and, since u is odd with respect to the space variable, we deduce (1.4) with $\varepsilon = \sup I_{x_*} > 0$. ■

Proof of Theorem 1.2
Lemma 3.1 still shows that $u \in C_b([0,+\infty[\times\mathbb{R}_*)$ is odd and non-increasing with respect to the space variable.
To prove Theorem 1.2, it suffices to show that there exists $0 < t_* < +\infty$ such that $u(t_*,\cdot)$ is discontinuous at $x = 0$, since Theorem 1.1 then states that the discontinuity persists a little while after t_* (because $u(t_*,\cdot)$ satisfies (1.3) by Lemma 3.1).
Defining $S(\lambda) = \frac{2^{1-\lambda}G_\lambda}{\lambda(1-\lambda)^2}$ as above, (1.5) gives $x_* > 0$ such that $u_0(x_*) + S(\lambda)x_*^{1-\lambda} =: -\rho < 0$. By Lemma 3.6, the generalized characteristic $x(t)$ originating from x_* is bounded from above by $x_* - \rho t$ for all $t \in I_{x_*}$, and its graph therefore crosses the axis $x = 0$ before the time $t = x_*/\rho$. This generalized characteristic thus cannot be defined on $[0,+\infty[$ and, by Item ii) in Lemma 3.5, we have $I_{x_*} = [0,t_*[$ with $t_* \le x_*/\rho < +\infty$ and $\lim_{t\to t_*^-} x(t) = 0$. If $y > 0$ then, for all $t < t_*$ close to t_*, we have $x(t) < y$ and, since $u(t,\cdot)$ is non-increasing on $\mathbb{R}_*^+$, we deduce $u(t,y) \le u(t,x(t)) \le -\rho$ by Lemma 3.6. Since u is continuous on $[0,+\infty[\times\mathbb{R}_*$, we can let $t \to t_*^-$ with $y > 0$ fixed to find $u(t_*,y) \le -\rho$. Hence, $\sup_{\mathbb{R}_*^+} u(t_*,\cdot) \le -\rho$ and, since $u(t_*,\cdot)$ is odd, this concludes the proof that it has a discontinuity at $x = 0$. ■

Remark 3.7 *Since $S(\lambda)$ used above has a finite limit as $\lambda \to 0$, the preceding proofs (and thus Theorems 1.1 and 1.2) are also valid with $\lambda = 0$, in which case (1.1) is reduced to $\partial_t u + \partial_x(\frac{u^2}{2}) + u = 0$ (see also*

Remark 4.2).

4 No creation of shock

In this section, we prove Theorem 1.4. The idea is to show that the approximations u^δ constructed by the splitting method stay Lipschitz continuous in space, with a Lipschitz constant not depending on δ. It is known that the hyperbolic parts of the splitting method (i.e. $\partial_t u^\delta + 2\partial_x(\frac{(u^\delta)^2}{2}) = 0$), have tendencies to make the Lipschitz constant of the solution explode; the key point is that, in this case, the fractal parts (i.e. $\partial_t u^\delta + 2g[u^\delta] = 0$) reduce the Lispchitz constant, and thus compensate for the explosion in the hyperbolic parts. This is what the following lemma states.

Lemma 4.1 *Let $\lambda \in]0,1[$ and $L > 0$ and $M > 0$ satisfy (1.7). Let $U_0 : \mathbb{R} \to \mathbb{R}$ be a smooth function which satisfies (1.3), and assume that U_0 is bounded by M and that U_0' is bounded by L. There exists $\delta_0 = \delta_0(\lambda, M, L) > 0$ such that, for all $\delta \le \delta_0$, if $U : [0, 2\delta] \times \mathbb{R} \to \mathbb{R}$ is constructed the following way:*

- *on $[0,\delta] \times \mathbb{R}$, U is the entropy solution to $\partial_t U + 2\partial_x(\frac{U^2}{2}) = 0$ with initial datum U_0,*
- *on $[\delta, 2\delta] \times \mathbb{R}$, U is the solution to $\partial_t U + 2g[U] = 0$ with initial datum $U(\delta, \cdot)$,*

then U satisfies

$$||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} \le \frac{L}{1-2L\delta} \qquad \text{for all } t \in [0, 2\delta], \tag{4.1}$$

and

$$||\partial_x U(2\delta,\cdot)||_{L^\infty(\mathbb{R})} \le L. \tag{4.2}$$

Proof of Lemma 4.1
On the time interval where U solves the Burgers equation, by the method of characteristics, one has $U(t, x_0 + 2tU_0(x_0)) = U_0(x_0)$ as long as $t < 1/2||U_0'||_{L^\infty(\mathbb{R})}$, and this relation completely defines U on $[0, 1/2||U_0'||_{L^\infty(\mathbb{R})}[\times\mathbb{R}$ (all the points in this set can be written as $(t, x_0 + 2tU_0(x_0))$ with $x_0 \in \mathbb{R}$). In particular, for all $t < 1/2L$ and all $x_0 \in \mathbb{R}$, we have

$$|\partial_x U(t, x_0 + 2tU_0(x_0))| = \left|\frac{U_0'(x_0)}{1 + 2tU_0'(x_0)}\right| \le \frac{L}{1-2tL}.$$

Hence, if $\delta < 1/2L$, the function U remains regular on $[0,\delta] \times \mathbb{R}$ and we have $||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} \le \frac{L}{1-2L\delta}$ for all $t \in [0,\delta]$, i.e. (4.1) is satisfied for $t \in [0,\delta]$.
For $t \in]\delta, 2\delta] \times \mathbb{R}$, we have $U(t,\cdot) = K(2(t-\delta),\cdot) * U(\delta,\cdot)$ and thus $\partial_x U(t,\cdot) = K(2(t-\delta),\cdot) * \partial_x U(\delta,\cdot)$. Since $K(s,\cdot)$ has a L^1 norm equal to 1 for all $s > 0$, we deduce that $||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} \le ||\partial_x U(\delta,\cdot)||_{L^\infty(\mathbb{R})}$ (i.e. the fractal equation does not increase the Lipschitz semi-norm), and (4.1) is therefore also satisfied for $t \in]\delta, 2\delta]$.

It remains to prove (4.2). As seen above, the fractal equation does not increase the Lipschitz semi-norm so that if $||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} \le L$ for some $t \in]\delta, 2\delta]$ then (4.2) is obvious. We can therefore assume that

$$||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} \ge L \qquad \text{for all } t \in]\delta, 2\delta]. \tag{4.3}$$

It has been shown in the proof of Lemma 3.1 that both the hyperbolic and the fractal equations preserve (1.3); hence, for all $t \in [0, 2\delta]$, $U(t,\cdot)$ satisfies (1.3). This means in particular that $\partial_x U(t,\cdot)$ is non-positive on $\mathbb{R}$ and has its absolute maximum value at $x = 0$. Let $\gamma(t) = ||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} = -\partial_x U(t, 0)$. On $]\delta, 2\delta] \times \mathbb{R}$ we have $\partial_t U = -2g[U]$; since g and ∂_x commute, this implies $\partial_t(\partial_x U) = -2g[\partial_x U]$, and in particular

$$\gamma'(t) = -2g[-\partial_x U(t,\cdot)](0) = 2G_\lambda \int_{\mathbb{R}} \frac{-\partial_x U(t,z) + \partial_x U(t,0)}{|z|^{1+\lambda}}\, dz \qquad \text{for all } t \in]\delta, 2\delta]. \tag{4.4}$$

For all $z \in \mathbb{R}$, since $-\partial_x U(t,0) = ||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})}$ we have $-\partial_x U(t,0) \geq -\partial_x U(t,z)$ and, therefore, for all $R > 0$,

$$\gamma'(t) \leq 2G_\lambda \int_{|z|\geq R} \frac{-\partial_x U(t,z) + \partial_x U(t,0)}{|z|^{1+\lambda}}\, dz. \tag{4.5}$$

Since $U(t,\cdot)$ satisfies (1.3) and is bounded by $||U_0||_{L^\infty(\mathbb{R})} \leq M$ (because the hyperbolic and the fractal equations do not increase the L^∞ norm), Lemma 5.2 in the appendix shows that $|\partial_x U(t,z)| \leq L/2$ for all $|z| \geq \frac{M}{L/2}$, which implies in particular, by (4.3),

$$-\partial_x U(t,z) = |\partial_x U(t,z)| \leq \frac{1}{2}||\partial_x U(t,\cdot)||_{L^\infty(\mathbb{R})} = -\frac{1}{2}\partial_x U(t,0) \qquad \text{for all } t \in]\delta, 2\delta] \text{ and all } |z| \geq \tfrac{2M}{L}.$$

Hence, taking $R = \frac{2M}{L}$ in (4.5) we find

$$\gamma'(t) \leq \partial_x U(t,0) G_\lambda \int_{|z|\geq 2M/L} \frac{dz}{|z|^{1+\lambda}} = -\gamma(t)\frac{2G_\lambda}{\lambda}\left(\frac{2M}{L}\right)^{-\lambda}.$$

Defining $P = P(\lambda, M, L) = \frac{2G_\lambda}{\lambda(2M)^\lambda} L^\lambda$, Gronwall's lemma then gives $\gamma(t) \leq e^{-P(t-\delta)}\gamma(\delta)$ for all $t \in]\delta, 2\delta]$. With $t = 2\delta$ and thanks to (4.1), this leads to

$$||\partial_x U(2\delta,\cdot)||_{L^\infty(\mathbb{R})} \leq \frac{e^{-P\delta}}{1-2L\delta} L.$$

The lemma is proved if we can show that, for δ small enough, we have $e^{-P\delta} \leq 1 - 2L\delta$. Since $e^{-P\delta} = 1 - P\delta + \mathcal{O}(\delta^2)$, this comes down to demanding that $P - \mathcal{O}(\delta) \geq 2L$ for δ small enough and, since (1.7) states that $P > 2L$, this concludes the proof. ■

The proof of Theorem 1.4 is now easy.

Proof of Theorem 1.4

Let $L > 0$ and $M > 0$ which satisfy (1.7) and $\delta_0 = \delta_0(\lambda, M, L)$ given by Lemma 4.1. Let u_0 satisfy (1.3), be bounded by M and with derivative bounded by L; for $\delta \leq \delta_0$, let u^δ be constructed by the splitting method as in Section 2.2. By the proof of Lemma 3.1, we know that, for all $t > 0$, $u^\delta(t,\cdot)$ satisfies (1.3), and it is quite obvious that u^δ stays bounded by M (because both fractal and hyperbolic equations do not increase the L^∞ norm).

On $]0,\delta] \times \mathbb{R}$, since we solve the fractal equation, we see that u^δ is smooth and that $\partial_x u^\delta$ is still bounded by L. From Lemma 4.1 with $U_0 = u^\delta(\delta,\cdot)$, we deduce that

$$||\partial_x u^\delta(t,\cdot)||_{L^\infty(\mathbb{R})} \leq \frac{L}{1-2L\delta} \quad \text{for } t \in [\delta, 3\delta], \tag{4.6}$$

and

$$||\partial_x u^\delta(t,\cdot)||_{L^\infty(\mathbb{R})} \leq L \quad \text{for } t = 3\delta. \tag{4.7}$$

This last estimate shows that $u^\delta(3\delta,\cdot)$, which is smooth since we have solved the fractal equation on $]2\delta, 3\delta] \times \mathbb{R}$, satisfies the assumptions on U_0 in Lemma 4.1; this allows to see that (4.6) is also satisfied for $t \in [3\delta, 5\delta]$ and that (4.7) is also satisfied for $t = 5\delta$, which allows in return to apply Lemma 4.1 with $U_0 = u^\delta(5\delta,\cdot)$, etc... . By induction, we conclude that (4.6) is satisfied for all $t \geq 0$ (it was clearly satisfied on $[0,\delta]$) and that (4.7) is satisfied for all $t = q\delta$ with q odd.

As $\delta \to 0$, u^δ converges to the entropy solution u to (1.1)-(1.2) in $C([0,T]; L^1_{\mathrm{loc}}(\mathbb{R}))$ for all $T > 0$; hence, by letting $\delta \to 0$ in (4.6) satisfied for all $t \geq 0$, we deduce that u is Lipschitz continuous with respect to the space variable and that $||\partial_x u(t,\cdot)||_{L^\infty(\mathbb{R})} \leq L$ for all $t \geq 0$. This implies that $g[u(t,\cdot)] \in C_b(\mathbb{R})$ is bounded independently of t and, since u is also a weak solution to (1.1), that $\partial_t u = -\partial_x(\frac{u^2}{2}) - g[u]$ in the distributional sense on $]0,+\infty[\times\mathbb{R}$. The time derivative of u is therefore bounded, and u belongs to $W^{1,\infty}(]0,+\infty[\times\mathbb{R})$. ■

Remark 4.2 *Since $\frac{G_\lambda}{\lambda}$ has a positive limit as $\lambda \to 0$, the preceding proof also works if $\lambda = 0$; in this case, Theorem 1.4 gives back known results on dissipative conservation laws (see e.g. [10]): under a smallness assumption on the Lipschitz constant of the initial data (and no assumption on its L^∞ norm), the solution to $\partial_t u + \partial_x(\frac{u^2}{2}) + u = 0$ does not develop shocks.*
Notice that, as explained in Remark 3.7, if the initial data is "too large" then shocks indeed occur in the solution to $\partial_t u + \partial_x(\frac{u^2}{2}) + u = 0$.

Remark 4.3 *Theorem 1.4 is also valid for $\lambda = 1$ (in which case (1.7) is only a condition on the L^∞ norm of the initial data). In this case, Formula (2.1) for g must be slightly modified: if $\lambda = 1$, then*

$$g[\varphi](x) = -G_1 \int_{|z|\leq 1} \frac{\varphi(x+z) - \varphi(x) - \varphi'(x)z}{|z|^2}\,dz - G_1 \int_{|z|\geq 1} \frac{\varphi(x+z) - \varphi(x)}{|z|^2}\,dz. \tag{4.8}$$

From the proof of Lemma 3.1, it is quite obvious that (1.1) preserves (1.3) even if $\lambda = 1$; we can therefore apply the technique in the proof of Lemma 4.1 to estimate $\gamma(t) = -\partial_x U(t,0)$ and, as 0 is an extremum of $\partial_x U(t,\cdot)$, we have $\partial_x(\partial_x U(t,\cdot))(0) = 0$; hence, when using (4.8) in (4.4), the new term involving $\varphi'(x)$ with $x = 0$ and $\varphi = -\partial_x U(t,\cdot)$ disappears and the proof of the estimates on $\partial_x U$ follows as in the case $\lambda < 1$.

5 Appendix

Lemma 5.1 *Let $f, h \in L^1(\mathbb{R})$ be even, non-negative on $\mathbb{R}$ and non-increasing on $\mathbb{R}^+$. Then, $f*h \in L^1(\mathbb{R})$ also satisfies these properties.*

Proof of Lemma 5.1
By definition of the convolution product, it is obvious that $f * h$ is non-negative and even. To show that $f * h$ is non-increasing on $\mathbb{R}^+$, let us first assume that $h \in C^1_c(\mathbb{R})$. In this case, we have

$$(f*h)'(x) = f*h'(x) = \int_{\mathbb{R}} f(x-y)h'(y)\,dy = \int_0^\infty f(x-y)h'(y)\,dy + \int_{-\infty}^0 f(x-y)h'(y)\,dy.$$

Since h is even, h' is odd and thus

$$(f*h)'(x) = \int_0^\infty f(x-y)h'(y)\,dy + \int_0^\infty f(x+y)h'(-y)\,dy = \int_0^\infty h'(y)(f(x-y) - f(x+y))\,dy. \tag{5.1}$$

Let $x \geq 0$.

- If $0 \leq y \leq x$ then $0 \leq x - y \leq x + y$ and, since f is non-increasing on $\mathbb{R}^+$, $f(x-y) \geq f(x+y)$.
- If $y \geq x$ then $0 \geq x - y \geq -y$ and, since f is non-decreasing on $\mathbb{R}^-$ (it is even and non-increasing on $\mathbb{R}^+$), $f(x-y) \geq f(-y) = f(y)$. But $0 \leq y \leq x + y$ and f is non-increasing on $\mathbb{R}^+$, so that $f(y) \geq f(x+y)$ and we conclude that $f(x-y) \geq f(x+y)$.

In either case, we get $f(x-y) - f(x+y) \geq 0$ for all $y \geq 0$. Since $h' \leq 0$ on $\mathbb{R}^+$ (because h is non-increasing on this interval), (5.1) shows that $(f*h)'(x) \leq 0$ for all $x \geq 0$, which concludes the proof that $f * h$ is non-increasing on $\mathbb{R}^+$ in the case where h is regular.

In the case where h is not regular, then it suffices to approximate it in $L^1(\mathbb{R})$ by regular functions h_n which are even, non-negative on $\mathbb{R}$ and non-increasing on $\mathbb{R}^+$. We then know that $f * h_n$ is non-increasing on $\mathbb{R}^+$ and converges, as $n \to \infty$, to $f * h$ in $L^1(\mathbb{R})$; up to a subsequence, the convergence holds a.e. on $\mathbb{R}$ and shows that $f * h$ is also non-increasing on $\mathbb{R}^+$ (or more precisely, if we do not know that $f * h$ is continuous, that it has an a.e. representative which is non-increasing on $\mathbb{R}^+$). ■

Lemma 5.2 *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be bounded by $M > 0$ and satisfy (1.3). Let $A > 0$. If $|x| \geq \frac{M}{A}$ then $|\varphi'(x)| \leq A$.*

Proof of Lemma 5.2
Since φ' is even, it is enough to prove the result for $x \geq \frac{M}{A}$. The slopes inequality applied to the convex function φ on $\mathbb{R}^+$ with the points $(0, \varphi(0)) = (0, 0)$ and $(x, \varphi(x))$ gives

$$\varphi'(x) \geq \frac{\varphi(x)}{x} \geq \frac{-M}{x}.$$

Since $\varphi' \leq 0$ on $\mathbb{R}^+_*$, we deduce that $|\varphi'(x)| = -\varphi'(x) \leq \frac{M}{x} \leq A$ and the proof is concluded. ■

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